Analog of Lie Algebra and Lie Group for Quantum Non-Hamiltonian Systems.

Vasily E.Tarasov,
Nuclear Physics Institute, Moscow State University,
119899 Moscow, RUSSIA
E-mail: TARASOV@THEORY.NPI.MSU.SU

arXiv:hep-th/9601063v1 12 Jan 1996

Abstract

In order to describe non-Hamiltonian (dissipative) systems in quantum theory we need to use non-Lie algebra such that commutators of this algebra generate Lie subalgebra. It was shown that classical connection between analytic group (Lie group) and Lie algebra, proved by Lie theorems, exists between analytic loop, commutant of which is associative subloop (group), and commutant Lie algebra (an algebra commutant of which is Lie subalgebra).

1 Introduction.

Quantum mechanics of Hamiltonian (non-dissipative) systems uses well known pair: Lie algebra and analytic group (Lie group). Elements of this pair are connected by Lie theorems. In order to describe non-Hamiltonian (dissipative) systems in quantum theory [1, 2, 3, 4] we need to use non-Lie algebra and analytic quasigroup (loop). This new pair consists of commutant Lie algebra (Valya algebra) and analytic commutant associative loop (Valya loop). A commutant Lie algebra (Valya algebra) is an algebra the commutators of which generate Lie subalgebra (an algebra such that commutant is a Lie subalgebra). Commutant Lie algebra can be defined by the following conditions $x^2 = 0$; $J(xy, zq, pl) = 0$, where $J(x, y, z) = (xy)z + (yz)x + (zx)y$ – Jacobian of elements x, y, z. Valya loop is a non-associative loop such that the commutant of this loop is associative subloop (group). It is known that a commutant of algebra is a subspace, which is generated by all commutators of the algebra. We proved that a tangent algebra of Valya loop is a commutant Lie algebra (Valya algebra) – it is analog of Lie theorem. It was shown that generalized Heisenberg-Weyl algebra, suggested by the author [1, 2, 3] to describe quantum non-Hamiltonian (dissipative) systems, is a commutant Lie algebra. As the other example of commutant Lie algebra, it was considered a generalized Poisson algebra [4] for differential 1-forms.

2 Generalized Heisenberg-Weyl algebra.

2.1. Definition of Generalized Heisenberg-Weyl Algebra.

Usual Heisenberg-Weyl algebra W_N is defined by basic elements $\{Q_i, P_i, I\}$ which are satisfied the following commutation relations

$$[Q_i, Q_j] = [P_i, P_j] = [Q_i, I] = [P_i, I] = 0 \ , \qquad [Q_i, P_j] = \imath\delta_{ij} I \tag{1}$$

Basis of generalized Heisenberg-Weyl algebra W_N^* consists of $2N + 2$ basic elements $\{Q_i, P_i, W, I\}$ which are satisfied the relations (1) and commutation relations [1, 2, 3, 4]:

$$[W, P_i] = \imath F_i(Q, P) \ , \qquad [W, Q_i] = [W, W] = [W, I] = 0 \tag{2}$$

In the relation (2) elements $F_i(Q, P)$ are functions (for example, polynomial) of basic elements Q_i, P_i. In the simplest case which is interesting in application, the relation (2) has the form $[W, P_i] = \imath\gamma_{ij} P_j$, where γ_{ij} – numbers. Generalized Heisenberg-Weyl algebra for this case we denote by LW_N^*. Let us note that physical meaning of elements $F_i(Q, P)$ are the following: $F_i(Q, P)$ is dissipative force (friction) which is acts on the system.

2.2. Identities for Jacobians.

Commutation relations (2) lead to the fact that some Jacobians of basics elements are not equal to zero [1, 14, 4]:

$$J[P_i, P_j, W] = \imath([F_i, P_j] - [F_j, P_i]) \ , \qquad J[Q_i, P_j, W] = -\imath[F_j, Q_i] \ , \tag{3}$$

where $J[x,y,z] = [[x,y],z]+[[y,z],x]+[[z,y],x]$ is Jacobian of elements x,y,z. The fact that Jacobians (3) are not equal to zero is a main distinction of non-Hamiltonian systems and it is characteristic property of suggested generalization of Heisenberg-Weyl algebra. In the consequence of relations (3) generalized Heisenberg-Weyl algebra W^*_N is non-Lie algebra.

2.3. General Element of Generalized Heisenberg-Weyl Algebra.

Let us consider general element Z of generalized Heisenberg-Weyl algebra W^*_N

$$Z = sI + x_iQ_i + y_iP_i + tW \ ,$$

where s, x_i, y_i, t – numbers. Commutator of general elements Z_1 and Z_2 has the form

$$[Z_1,Z_2] = \imath s_3 I + \imath l_i F_i(Q,P) \ , \quad where \quad s_3 = x^1_i y^2_i - x^2_i y^1_i \quad l_i = t^1 y^2_i - t^2 y^1_i \ .$$

Jacobian of general elements Z_1, Z_2 and Z_3 has the form

$$J[Z_1,Z_2,Z_3] = is_{ij}[Q_i,F_j] + it_{ij}([F_i,P_j]-[F_j,P_i]) \ ,$$

$$where \quad s_{ij} = (x^1_i y^2_j - x^2_i y^1_j)t^3 + (x^1_i y^3_j - x^3_i y^1_j)t^2 + (x^2_i y^3_j - x^3_i y^2_j)t^1$$

$$t_{ij} = 2(y^2_i y^3_j t^1 + y^3_i y^1_j t^2 + y^1_i y^2_j t^3) \quad .$$

In the simplest case with $F_i(Q,P) = \gamma_{ij}P_j$ the commutator of general elements Z_1 and Z_2 has the form

$$[Z_1,Z_2] = \imath Z_3 \ , \qquad where \quad s_3 = x^1_i y^2_i - x^2_i y^1_i \quad x^3_i = 0 \quad y^3_i = t^1 y^2_i - t^2 y^1_i \quad t^3 = 0 \ ,$$

and Jacobian of elements Z_1, Z_2 and Z_3 is

$$J[Z_1,Z_2,Z_3] = Z_4 \ , \qquad where \quad s_4 = -s_{ij}\gamma_{ij} \qquad x^4_i = y^4_i = t^4 = 0 \quad .$$

3 Commutant Lie Algebra.

3.1. Let us introduce

DEFINITION 1. An algebra B will be called **Valya algebra**, if multiplicative operation is satisfied the following conditions

1) Anti-symmetric identity: $xx = 0$;

2) Soft Jacobi identity: $J(xy,zp,ql) = 0$,

where $J(x,y,z) = (xy)z + (yz)x + (zx)y$.

It is easy to see that Lie algebra, which is defined by the conditions $x^2 = 0$, $J(x,y,z) = 0$, is Valya algebra. A binary Lie algebra, the basic operation of which satisfies the conditions $x^2 = 0$, $J(x,y,xy) = 0$, is Valya algebra.

Let A be a non-associative algebra. Let us introduce a binary operation: $[x,y] = xy - yx$ which is called commutator. As the result we derive anti-commutative algebra $A^{(-)}$, which is associated with the algebra A. For this algebra anti-symmetric and soft Jacobi conditions are realized in the form

1) Anti-symmetric identity: $[x,y] = 0$;

2) Soft Jacobi identity: $J[[x,y],[z,p],[q,l]] = 0$.

Let us denote by $g = A^{(-)}$ an anti-commutative algebra $A^{(-)}$, which is associated with non-associative algebra A. It is known the following

DEFINITION 2. A **commutant** of the algebra g is a subspace $[g,g] = g'$, which is generated by all commutators $[x,y]$, where $x,y \in g$.

DEFINITION 3. A **divisible commutants** $g^{(k)}$ $k = 0,1,2,...$ of the algebra g are defined by the rule $g^{(0)} = g \quad g^{(k+1)} = (g^{(k)})'$

It is easy to prove the following

PROPOSITION

1. A commutant of algebra g is ideal, but a commutant $g^{(2)}$ of commutant $g^{(1)}$ is not ideal of this algebra g.
2. A divisible commutant $g^{(k+1)}$ is ideal of the algebra $g^{(k)}$, but is not ideal of algebra $g^{(l)}$, where $l < k$.

3. If Jacobi identity is satisfied then a divisible commutant $g^{(k+1)}$ is ideal of the algebra $g^{(l)}$, where $l < k$.

Let us introduce a new definition

DEFINITION 4. An algebra $g = A^{(-)}$ will be called a **commutant Lie algebra**, if the commutators of this algebra generate subalgebra which is Lie algebra, or if the commutant g' of the algebra g is Lie subalgebra.

It is easy to prove the following

PROPOSITION. A commutant Lie algebra is Valya algebra.

PROOF:

Let us consider elements z_k of commutant Lie algebra $A^{(-)}$, where $k = 1, 2, ..., 6$. Using the definition of Lie algebra and definition of commutant Lie algebra the following conditions are satisfied

1) Anti-symmetric identity: $[z_k, z_l] = 0$;

2) Soft Jacobi identity: $J[[z_1, z_2], [z_3, z_4], [z_5, z_6]] = 0$

3.2. Let us introduce the following new definition

DEFINITION 5. An algebra A will be called a **commutant associative algebra**, if associator of commutators is equal to zero, that is the basic operation satisfies the following conditions

$$(xy - yx, zp - pz, ql - lq) = 0 \ , \qquad (4)$$

$$or \ \ (xy, zp, ql) + (xy, pz, lq) + (yx, zp, lq) + (yx, pz, ql) -$$

$$-(yx, zp, ql) - (xy, pz, ql) - (xy, zp, lq) - (yx, pz, lq) = 0 \ ,$$

where $(x, y, z) = (xy)z - x(yz)$ – associator of algebra elements.

THEOREM.

Let A be a commutant associative algebra. An anti-commutative algebra $A^{(-)}$ of this commutant associative algebra A is a commutant Lie algebra.

PROOF:

Let us use the connection between Jacobian and associator in the form

$$J[x, y, z] = (x, y, z) + (y, z, x) + (z, x, y) - (z, y, x) - (y, x, z) - (x, z, y)$$

If we consider the Jacobian of commutators

$$J[[x, y], [z, p], [q, l]] = J[xy - yx, zp - pz, ql - lq]$$

and definition condition (4) of commutant associative algebra then we derive soft Jacobi identity and anti-symmetric identity for the algebra $A^{(-)}$.

3.3. It is easy to prove the following

PROPOSITION.

A generalized Heisenberg-Weyl algebra W^*_N is a commutant Lie algebra (Valya algebra), which is non-Lie algebra.

PROPOSITION.

A generalized Heisenberg-Weyl algebra LW^*_N, which is defined by the relations (1-2) with $F_i(Q, P) = \gamma_{ij} P_j$, satisfies not only soft Jacobi identity but the following conditions are satisfied

$$1) \ J[[Z_1, Z_2], [Z_3, Z_4], Z_5] = 0 \ ; \ \ 2) \ [J[Z_1, Z_2, Z_3], Z_4] = 0 \ ; \ \ 3) \ [[Z_1, Z_2], [Z_3, Z_4]] = 0 \qquad (5)$$

Note that first identity and soft Jacobi identity are the consequence of third condition. It is easy to prove the following

PROPOSITIONS.

1. Heisenberg-Weyl algebra W_N is an ideal of generalized Heisenberg-Weyl algebra W^*_N.
2. A commutant of generalized Heisenberg-Weyl algebra W^*_N is a subalgebra of Heisenberg-Weyl algebra W_N.
3. Maximal ideal of generalized Heisenberg-Weyl algebra W^*_N, which is Lie algebra, is Heisenberg-Weyl algebra W_N.
4. Annulator of generalized Heisenberg-Weyl algebra W^*_N is an annulator of Heisenberg-Weyl algebra W_N { Z : Z=sI }.

4 Loops with Associative Commutants.

4.1. Loop with Inverse Elements and Loop Commutant.

Analytic loop – non-associative generalization of analytic group (Lie group) – first consider by A.I. Malcev [5], see also [6, 7, 8, 9].

Let G be a**loop with inverse elements** [9], that is a set G with binary operation ($\circ$) such that the following conditions are satisfied

1) There exists a fixed element $e \in G$ such that $x \circ e = e \circ x = x$ for all x in G;

2) For each element $a \in G$ there exists unique inverse element a^{-1} in G such that: $a \circ a^{-1} = a^{-1} \circ a = e$

3) For all elements $a, b \in G$ the following identity are satisfied: $(b \circ a) \circ a^{-1} = a^{-1} \circ (a \circ b) = b$

Note that all elements a, b in this loop G satisfy the following important identity [9]:

$$(a \circ b)^{-1} = b^{-1} \circ a^{-1}$$

It is known that a commutator of the loop with inverse element can be defined. A **commutator** of the elements $x, y \in G$ is a element z in G such that

$$z = [x, y] = (x \circ y) \circ (y \circ x)^{-1} = (x \circ y) \circ (x^{-1} \circ y^{-1})$$

DEFINITION 6. A **commutant of loop** G is a set $G^{(-)}$ of elements z in G, which can be represented in the form $z = z_1 z_2 ... z_m$, where z_i are commutators of elements $x_i, y_i \in G$.

4.2. Tangent Algebra of Analytic Loop.

Following [5] we shall call an **analytic loop** an analytic manifold, such that the binary operation on the manifold satisfies loop structure conditions and is analytic operation [5].

A **tangent algebra** g of local analytic loop G is a tangent space $T_e(G)$ together with binary and ternary operations $[\ ,\ ]$ and $<\ ,\ ,\ >$ which are defined in the following form [7]. Let $\alpha(t),\ \beta(t),\ \gamma(t)$ – smooth curves in a loop G, going through the point e $\alpha(0) = \beta(0) = \gamma(0) = e$ and having tangent vectors $\alpha'(0) = \xi,\ \beta'(t) = \eta,\ \gamma'(t) = \zeta$. Then

$$(\beta(t)\alpha(t))\backslash(\alpha(t)\beta(t)) = t^2[\xi, \eta] + o(t^2) \tag{6}$$

$$(\alpha(t)(\beta(t)\gamma(t)))\backslash((\alpha(t)\beta(t))\gamma(t)) = t^3 < \xi, \eta, \zeta > + o(t^3) \tag{7}$$

Using local coordinate on analytic loop G in the neighborhood of the point e the production $z = x \circ y$ can be expanded in the Teylor series $z_i = \mu_i(x, y)$ in the form

$$\mu_i(x, y) = x_i + y_i + a^i_{jk} x_j y_j + b^i_{jkl} x_j x_k y_l + c^i_{jkl} x_j y_k y_l + ... \tag{8}$$

Binary and ternary operations have the form

$$[\xi, \eta]_i = u^i_{jk} \xi^j \eta^k \qquad < \xi, \eta, \zeta >_i = v^i_{jkl} \xi^j \eta^k \zeta^l \tag{9}$$

$$where\ \ u^i_{jk} = a^i_{jk} - a^i_{kj}\ , \quad v^i_{jkl} = 2b^i_{jkl} - 2c^i_{jkl} + \frac{1}{4} u^i_{ml} v^m_{jk} - \frac{1}{4} v^i_{jm} v^m_{kl} \tag{10}$$

That is tangent space can be equipped with composition laws (9) and the resulting binary-ternary algebra is called a tangent loop algebra. Note, that if analytic loop is associative, i.e. is Lie algebra, then ternary operation is equivalent to zero $< \xi, \eta, \zeta >= 0$ for all $\xi,\ \eta,\ \zeta \in g$. If analytic loop is binary associative loop (alternative loop) [7] then the ternary operation is completely defined by the binary operation $< \xi, \eta, \zeta >= (1/6)J[\xi, \eta, \zeta]$. In normal coordinates of arbitrary local analytic loop we obtain $x \circ y = x + y + (1/2)xy + ...$, where xy – binary anti-symmetric ($xx = 0$ or $xy = -yx$) operation for elements of tangent loop algebra. Let us prove that a loop commutator can be written by

$$[x, y] \equiv (x \circ y) \circ (x \circ y)^{-1} = xy + ... \tag{11}$$

where dots denote combination of degree more than 3. Really

$$x \circ y = x + y + (1/2)xy + ... , \qquad (x \circ y)^{-1} = -x - y - (1/2)xy - ...$$

$$[x,y] = (x \circ y) \circ (x \circ y)^{-1} = x + y + (1/2)xy -$$

$$-x - y - (1/2)xy + (1/2)(x+y)(-x-y) + ... = -(1/2)(xx - yy - 2yx) \ .$$

For anti-symmetric operation ($xx = 0$ and $xy = -yx$) we derive the formula (11).

4.4. Tangent Algebra of Commutant Associative Loop.

A **commutator** of the elements $x, y \in G$ is an element $(x \circ y) \circ (y \circ x)^{-1}$. Let us remind that **commutant** $G^{(-)}$ of a loop G is a subloop, which is generated by commutators of this loop $(x \circ y) \circ (y \circ x)^{-1}$, where $x, y \in G$.

Let us introduce new definition

DEFINITION 7. A **commutant associative loop (Valya loop)** is a loop with inverse elements the commutant of which (a set of commutators) is associative subloop (group).

Analog of Lie theorem is formulated in the following form

THEOREM.

a. A tangent algebra of local analytic commutant associative loop (Valya loop) is a commutant Lie algebra (Valya algebra).

b. An arbitrary Valya (commutant Lie) algebra is a tangent algebra of some local analytic Valya (commutant associative) loop.

PROOF:

a. Let us consider an arbitrary element of a loop commutant $G^{(-)}$. According to the definition an element of the commutant can be represented in the form $z_1 z_2 ... z_m$, where z_i is a commutator of elements x_i y_i of the loop G. If a set G is a commutant associative loop, then the product $g_1(t_1)g_2(t_2)...g_m(t_m)$, where $g_i(t_i)$ are arbitrary one-parameter subgroups with tangent vectors z_i, is not depends on brackets order for all small t_i. Therefore, a subalgebra of tangent loop algebra, which is generated by elements z_i, is Lie algebra, that is the tangent algebra is commutant Lie algebra (Valya algebra).

b. Let us introduce a normal coordinates. Note, that one-parameter subgroups $g_i(t_i)$ generate a local associative subloop. It is easy to see that in normal coordinates we have Teylor power series, which allows to derive a loop from an algebra.

Note that the proof of part (b) in general case is open question.

5 Realization of Commutant Lie Algebra as Algebra of 1-Forms.

5.1. Poisson Algebra for Differential 1-Forms.

It is known that Poisson brackets can be defined for non-closed differential 1-forms on symplectic manifold [10]. Poisson brackets for two 1-forms $\alpha = a_k(z)dz^k$ and $\beta = b_k(z)dz^k$ on symplectic manifold (M, ω), is 1-form (α, β), defined by

$$(\alpha, \beta) \ = \ d\Psi(\alpha, \beta) \ + \ \Psi(d\alpha, \beta) \ + \ \Psi(\alpha, d\beta) \tag{12}$$

$$where \ \ \Psi(\alpha, \beta) \ = \ \omega(X_\alpha, X_\beta) \ = \ \Psi^{kl} a_k b_l \ ; \tag{13}$$

and X_α − vector field, which corresponds to 1-form α by the rule: $i(X_\alpha)\omega = \alpha$; ω - closed ($d\omega = 0$) non-degenerate 2-form is called a symplectic form; i − internal multiplication of vector fields and differential forms [11]; Ψ is cosymplectic structure and Ψ^{kl} is 2-tensor, which is a matrix inverse to matrix of symplectic form and satisfies the following conditions [10]:

a) Skew-symmetry: $\Psi^{kl} = \Psi^{lk}$

b) Zero Schouten brackets:

$$[\Psi, \Psi]^{slk} = \Psi^{sm} \partial_m \Psi^{lk} + \Psi^{lm} \partial_m \Psi^{ks} + \Psi^{km} \partial_m \Psi^{sl} = 0$$

If this bilinear operation "Poisson bracket" is defined on the space of 1-forms $\Lambda^1(M)$, then a manifold M is called Poisson manifold, and the space $\Lambda^1(M)$ with this operation − Poisson algebra P_1. Poisson algebra P_1 is Lie algebra. It is cased by skew-symmetry $(\alpha, \beta) = -(\beta, \alpha)$ and Jacobi identity:

$$J(\alpha, \beta, \gamma) = ((\alpha, \beta), \gamma) + ((\beta, \gamma)\alpha) + ((\gamma, \alpha), \beta) \ = \ 0$$

5.2. Non-Closed Forms and Non-Hamiltonian Systems.

It is known that a **physical system** in classical mechanics is a vector field X on the symplectic manifold (M^{2n}, ω).

Non-Hamiltonian and dissipative properties of the system are connected with properties of non-closed differential form which can be derived from correspondent vector field.

Following [11] we shall call **Hamiltonian system** on the symplectic manifold (M^{2n}, ω) a vector field X such that differential 1-form $i_X\omega$ is closed.

If the form $i_X\omega$ is exact form, then **Hamiltonian** of the system X is a function H on M^{2n}, which satisfies the following equation $i_X\omega = -dH$.

Let us introduce the following obvious definition:

DEFINITION 8. **Non-Hamiltonian (dissipative) system** on symplectic manifold (M^{2n}, ω) is a vector fields X such that the differential 1-form $i_X\omega$ is non-closed.

5.3. Generalized Poisson Algebra for Differential 1-Forms.

In order to describe non-Hamiltonian systems we suggest [4] to generalize Poisson algebra P_1. Let us define a new operation on the space $\Lambda^1(M)$.

DEFINITION 9. A **generalized Poisson bracket** of two 1-forms α and β is a closed 1-form (α, β), defined by

$$[\alpha,\beta] \ = \ d\ \Psi(\alpha,\beta) \ = \ d\ \omega(X_\alpha, X_\beta) \tag{14}$$

It is easy to see that Jacobi identity for non-closed 1-forms is not satisfied.

$$J[\alpha,\beta,\gamma] = [[\alpha,\beta],\gamma] + [[\beta,\gamma]\alpha] + [[\gamma,\alpha],\beta] \ \not\equiv \ 0$$

Therefore generalized Poisson algebra P_1^* is non-Lie algebra. Note that Jacobi identity is satisfied for closed 1-forms. So closed 1-forms define Lie algebra, which is Poisson algebra P_1 of closed 1-forms. In the result, generalized Poisson algebra P_1^* contains a subalgebra which is Poisson algebra. Generalized Poisson bracket of two non-closed 1-forms is closed 1-form, so the subalgebra P_1 is an ideal of the algebra P_1^* and exact diagram exists: $0 \to P_1 \to P_1^* \to P_1^*/P_1 \to 0$.

Since generalized Poisson bracket is a closed 1-form, the following proposition easy to prove

PROPOSITION.

A generalized Poisson algebra is a commutant Lie algebra (Valya algebra), that is a generalized Poisson bracket (14) satisfies anti-symmetric identity $[\alpha, \alpha] = 0$ and soft Jacobi identity:

$$J[[\alpha,\beta],[\gamma,\delta],[\mu,\nu]] = 0$$

Finally, the commutant Lie algebra, which is non-Lie algebra, can be naturally defined in the space of differential 1-forms $\Lambda^1(M)$ on symplectic manifold M. This algebra is generalization of Lie-Poisson algebra of closed 1-forms and contains this Lie algebra as subalgebra (ideal).

An obvious application of Valya algebra and Valya loop is a
quantum description of non-Hamiltonian (dissipative) systems.

Let us note in the conclusion that non-Hamiltonian (dissipative) quantum theory has a broad range of application, see the references in [2, 3]. For example, consistent theory of bosonic string in affine-metric curved space is non-Hamiltonian quantum theory [12, 14, 13].

References

[1] V.E. Tarasov: *Phys. Atom. Nucl.*, 56 N.11 (1993) 1608–1612.

[2] V.E. Tarasov: *Theor. Math. Phys.*, 100 N.3 (1994) 1100–1116.

[3] V.E. Tarasov: *Preprint ICTP*, IC-94-192, 1994. (hep-th/9410025)

[4] V.E. Tarasov: *Proc. IX Intern. Workshop on High Energy Physics and Quantum Field Theory, Zvenigorod, Russia, September 1994* (Moscow: Univer.Press, 1995. p.332-339) (hep-th/9506053)

[5] A.I. Malcev "Analytic loop." Matem. Sborn., 36, N.3 (1955), 569; "Selected works. Vol 1. Classical algebra." Moscow: Nauka, 1976, 340-345. (Russian)

[6] V.D. Belousov "Foundation of the theory of quasigroups and loops." Moscow: Nauka, 1967. (Russian)

[7] L.V.Sabinin, P.O.Miheev, "Theory of smooth Bol loop." Moscow: UDN Press, 1985, 80p.

[8] R.H.Bruck "A survey of binary systems." Berlin e.a.: Springer, 1958 (3ed.–1971), 185p.

[9] A.G. Kurosh "General algebra. Lectures of 1969-1970 educational year." Moscow: Nauka, 1974.–160p. (Russian)

[10] M.V. Karasev and V.P. Maslov "Nonlinear Poisson Brackets. Geometry and Quantization." Moscow: Nauka, 1990.(Russian)

[11] C. Godbillon "Geometrie Differentielle et mecanicue analytique." Paris: Herman, 1969.

[12] V.E. Tarasov: *Phys. Lett. B*, **323**, (1994) 296–304.

[13] V.E. Tarasov: *Mod. Phys. Lett. A*, **9** (1994) 2411–2419.

[14] V.E. Tarasov: *Theor. Math. Phys.*, 101 N.1 (1994) 1184–1190.